



Thermodynamics of no-tension materials

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Abstract

This paper presents a constitutive equation for no-tension materials in the presence of thermal expansion that accounts for the temperature-dependence of their material's constants. Specifically, assuming that the symmetric part of the displacement gradient minus the thermal dilatation is small, an explicit expression is given for stress from which the free energy, internal energy, entropy and enthalpy are obtained. Then, the equilibrium energy equations of a no-tension solid are presented, and we observe that, under further suitable hypotheses, thermo-mechanical uncoupling occurs. Finally, the work lost during an adiabatic process by two spherical containers made of a linear elastic and a no-tension material are compared. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

With the aim of modelling the behaviour of masonry structures, the constitutive equation of materials not withstanding tension has been studied by many authors under isothermal conditions (Del Piero, 1989; Panzeca and Polizzotto, 1988). The infinitesimal strain is assumed to be the sum of a positive semi-definite inelastic part and an elastic part on which the stress, negative semi-definite, depends linearly. Moreover, stress and inelastic strain, which can be interpreted as fracture strain, are orthogonal. Thus, one obtains a non-linear hyperelastic material, called *masonry-like or no-tension material*.

The existence and the uniqueness of the solution to this equation have been proved and the solution itself calculated explicitly in the isotropic case. More recently, suitable numerical techniques have been studied which allow application of the constitutive equation to solution of

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the equilibrium problem of masonry solids through the finite element method (Lucchesi et al., 1994, 1995, 1996). These techniques have yielded sound results mainly in the study of arches and vaults.

However, there are many engineering problems in which the presence of thermal dilatation must be accounted for; consider for example, the influence of thermal variations on stress fields in masonry bridges (Guidi, 1906), or the thermo-mechanical behaviour of the refractory materials used in the iron and steel industry (Kienow and Hennieke, 1966), and finally, geological problems connected with the presence of volcanic calderas such as that of Pozzuoli (Como and Lembo, 1989). In many such cases, the thermal variation during the thermo-mechanical process under examination is so high that the dependence of the material constants on temperature cannot be ignored.

In what follows, we develop a constitutive equation for isotropic no-tension materials under non-isothermal conditions. The theory presented here has allowed the study of numerical techniques for solution to the equilibrium problem of masonry-like solids in the presence of thermal loads via the finite element method (Padovani et al., 1999).

In Section 2, we present our assumptions and then set forth a non linear constitutive equation for no-tension materials under non-isothermal conditions. In view of the target applications, no limitations are placed on the range of temperature variation, but we do assume that the strain $\mathbf{E} - \beta(\theta)\mathbf{I}$ is small, with \mathbf{E} the symmetric part of the displacement gradient, θ the absolute temperature, $\beta(\theta)$ the thermal expansion and \mathbf{I} the identity tensor. We then suppose that $\mathbf{E} - \beta(\theta)\mathbf{I}$ is the sum of an elastic part \mathbf{E}^e and an inelastic part \mathbf{E}^a , orthogonal to the stress \mathbf{T} and positive semi-definite, that \mathbf{T} is negative semi-definite and depends linearly and isotropically on \mathbf{E}^e . We thereby obtain a non-linear elastic material that in the absence of thermal variation conforms to the masonry-like materials presented in Refs. (Del Piero, 1989; Panzeca and Polizzotto, 1988). Once the stress is explicitly calculated as function of \mathbf{E} and θ , we can then deduce the free energy, internal energy, entropy and enthalpy, define the specific heat at constant strain and the specific heat at constant stress, and finally compare them. By assuming the classical Fourier hypothesis for heat flux, the material presented here is characterised completely by five functions of the temperature: Young's modulus, Poisson's ratio, thermal expansion, conductivity and specific heat. In fact, when these material functions are known, the thermodynamic potentials (and consequently the thermo-mechanical behaviour) of the material are determined. At this point, once the energy equation has been obtained, we are in a position to write the basic equations of the thermoelastic theory for no-tension materials. Just as in the linear elastic case, these equations are: the strain-displacement relation, the equilibrium equation, the constitutive equations for stress and heat flux, and the equilibrium energy equation. The system we obtain is coupled because the temperature coefficient and the coefficient of the derivative of temperature with respect to time in the energy equation depend on strain and strain rate. However, if we assume that the displacement gradient, thermal expansion and its derivative with respect to temperature, strain rate and temperature rate are small, then the thermoelastic equilibrium equations are uncoupled and can be integrated separately. Treatment of the theory is fully three-dimensional; Appendix A summarises the results for plane stress.

Finally, Section 3 is devoted to consideration of two spherical containers which undergo an adiabatic process, under the hypothesis of thermo-mechanical uncoupling. The two containers, \mathcal{S}^e and \mathcal{S}^m , are made of a linear elastic and a no-tension material, respectively, and are subjected to the action of two uniform radial pressures acting on both the inner and outer boundaries and to a steady temperature distribution. The values of temperatures and pressures on the inner and outer boundary are chosen in such a way as the initial states of \mathcal{S}^e and \mathcal{S}^m coincide. At the end of the adiabatic process, \mathcal{S}^e and \mathcal{S}^m reach the equilibrium temperatures $\bar{\theta}_e$, and $\bar{\theta}_m$, respectively, as calculated explicitly using the first principle of thermodynamics. Finally, the variation in entropy during the process is calculated for both

containers, thus allowing for comparison of the work lost (Bailyn, 1994) in the transition from the initial to final state.

2. No-tension materials under non-isothermal conditions

The aim of this section is to formulate a constitutive theory and set forth the basic equations for the thermoelastic equilibrium of isotropic no-tension solids.

Let \mathcal{V} be a three-dimension linear space, Lin be the space of second order tensors equipped with the inner product $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B})$, $\mathbf{A}, \mathbf{B} \in \text{Lin}$, with \mathbf{A}^T the transpose of \mathbf{A} and tr the trace functional. For Sym , the space of all symmetric tensors, we denote by Sym^+ and Sym^- the convex cones of Sym constituted by positive semi-definite and negative semi-definite tensors, respectively.

For \mathbf{u} the displacement field, let $\mathbf{E} = (1/2)(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, the symmetric part of the displacement gradient, \mathbf{T} be the Cauchy stress tensor, $\theta \in [\theta_1, \theta_2]$, with $\theta_1 > 0$, be the *absolute temperature* and $\theta_0 \in [\theta_1, \theta_2]$ the reference temperature. We suppose that the thermal dilatation due to the temperature variation $\theta - \theta_0$ is the spherical tensor $\beta(\theta)\mathbf{I}$, with \mathbf{I} the second-order identity tensor and $\beta(\theta)$ a material function of the temperature called *thermal expansion*, with $\beta(\theta_0) = 0$.

We assume that there exists $\delta \in [0, 1)$ such that

$$\|\mathbf{E} - \beta(\theta)\mathbf{I}\| \leq \delta, \quad \text{for each } \theta \in [\theta_1, \theta_2], \tag{1}$$

where $\|\cdot\|$ is the norm induced by the scalar product in Lin , $\|\mathbf{A}\| = (\mathbf{A} \cdot \mathbf{A})^{1/2}$. Condition (1) is equivalent to requiring that the norm of the deviatoric part of \mathbf{E} and the scalar $\beta(\theta) - 1/3 \text{tr}(\mathbf{E})$ be $O(\delta)$.¹

2.1. Constitutive equations for the stress

Let $E(\theta)$ and $\nu(\theta)$ be temperature-dependent material functions, such that

$$E(\theta) > 0, \quad 0 \leq \nu(\theta) < \frac{1}{2}, \quad \text{for each } \theta \in [\theta_1, \theta_2], \tag{2}$$

and let us set

$$\gamma(\theta) = \frac{\nu(\theta)}{1 - 2\nu(\theta)}. \tag{3}$$

Generalising the constitutive equation of no-tension materials, we set forth a non-linear elastic constitutive equation which associates a negative semi-definite stress \mathbf{T} to each strain $\mathbf{E} - \beta(\theta)\mathbf{I}$.

We assume that

$$\mathbf{E} - \beta(\theta)\mathbf{I} = \mathbf{E}^c + \mathbf{E}^a, \tag{4}$$

$$\mathbf{T} = \mathbb{C}(\theta)[\mathbf{E}^c], \tag{5}$$

$$\mathbf{T} \cdot \mathbf{E}^a = 0, \tag{6}$$

¹ Given a mapping B from a neighbourhood of 0 in \mathbb{R} into a vector space with norm $\|\cdot\|$, we write $B(\delta) = O(\delta)$ if there exist $k > 0$ and $k' > 0$ such that $\|B(\delta)\| < k|\delta|$ whenever $|\delta| < k'$.

$$\mathbf{T} \in \text{Sym}^-, \quad (7)$$

$$\mathbf{E}^a \in \text{Sym}^+, \quad (8)$$

where $\mathbb{C}(\theta)$ is the definite positive fourth-order tensor

$$\mathbb{C}(\theta) = \frac{E(\theta)}{1 + \nu(\theta)} (\mathbb{I} + \gamma(\theta) \mathbf{I} \otimes \mathbf{I}), \quad (9)$$

with \mathbb{I} the fourth-order identity tensor and $\mathbf{I} \otimes \mathbf{I}$ defined by $\mathbf{I} \otimes \mathbf{I}[\mathbf{B}] = (\text{tr} \mathbf{B}) \mathbf{I}$, $\mathbf{B} \in \text{Lin}$. \mathbf{E}^e and \mathbf{E}^a are the elastic part and the inelastic part of $\mathbf{E} - \beta(\theta) \mathbf{I}$, respectively; \mathbf{E}^a can be also called fracture strain because one can expect fractures in the regions where \mathbf{E}^a is non-zero. It can be proved that for each (\mathbf{E}, θ) belonging to $\text{Sym} \times [\theta_1, \theta_2]$, \mathbf{T} is the solution of constitutive equations (4)–(8), if and only if $\mathbf{T} \in \text{Sym}^-$ and satisfies the variational inequality

$$(\mathbf{T} - \mathbf{T}^*) \cdot (\mathbf{E} - \beta(\theta) \mathbf{I} - \mathbb{C}(\theta)^{-1}[\mathbf{T}]) \geq 0, \quad \text{for each } \mathbf{T}^* \in \text{Sym}^-. \quad (10)$$

Inequality (10) means that \mathbf{T} is the projection of $\mathbb{C}(\theta)[\mathbf{E} - \beta(\theta) \mathbf{I}]$ onto the closed and convex cone Sym^- , with respect to the inner product $\mathbf{R} \circ \mathbf{S} = \mathbf{R} \cdot \mathbb{C}(\theta)^{-1}[\mathbf{S}]$ in Sym . Moreover, $\mathbf{E}^a = \mathbf{E} - \beta(\theta) \mathbf{I} - \mathbb{C}(\theta)^{-1}[\mathbf{T}]$ belongs to the normal cone $\mathcal{N}(\mathbf{T})$ to Sym^- at \mathbf{T} , with

$$\mathcal{N}(\mathbf{T}) = \begin{cases} \{\mathbf{N} \in \text{Sym} | (\mathbf{T} - \mathbf{T}^*) \cdot \mathbf{N} \geq 0, \forall \mathbf{T}^* \in \text{Sym}^-\}, & \text{if } \mathbf{T} \in \partial \text{Sym}^-, \\ \{\mathbf{0}\}, & \text{if } \mathbf{T} \in (\text{Sym}^-)^\circ, \end{cases} \quad (11)$$

where ∂Sym^- and $(\text{Sym}^-)^\circ$, respectively, denote the boundary and interior of Sym^- . It is an easy matter to show that \mathbf{N} belongs to $\mathcal{N}(\mathbf{T})$, if and only if $\mathbf{N} \in \text{Sym}^+$ and $\mathbf{N} \cdot \mathbf{T} = 0$. Moreover, if \mathbf{N} belongs to $\mathcal{N}(\mathbf{T})$, \mathbf{T} and \mathbf{N} are coaxial, and for t_i and n_i ($i = 1, 2, 3$), the eigenvalues of \mathbf{T} and \mathbf{N} , we have

$$t_1 n_1 = t_2 n_2 = t_3 n_3 = 0. \quad (12)$$

Let e_1, e_2, e_3 with $e_1 \leq e_2 \leq e_3$ be the eigenvalues of \mathbf{E} , and $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$, the corresponding eigenvectors. Let us put $\mathbf{O}_1 = \mathbf{f}_1 \otimes \mathbf{f}_1$, $\mathbf{O}_2 = \mathbf{f}_2 \otimes \mathbf{f}_2$, $\mathbf{O}_3 = \mathbf{f}_3 \otimes \mathbf{f}_3$. Since \mathbf{T} and \mathbf{E}^e are coaxial in view of the isotropy of $\mathbb{C}(\theta)$, we can conclude that $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ are also an eigenvectors basis for \mathbf{T} , \mathbf{E}^a and \mathbf{E}^e . Due to this, from Eqs. (4)–(8) we obtain a linear complementarity problem that can be solved by considering the following subsets of $\text{Sym} \times [\theta_1, \theta_2]$

$$\mathcal{A}_1 = \{(\mathbf{E}, \theta) | e_3 - \beta(\theta) + \gamma(\theta)(\text{tr} \mathbf{E} - 3\beta(\theta)) \leq 0\}, \quad (13)$$

$$\mathcal{A}_2 = \{(\mathbf{E}, \theta) | e_1 - \beta(\theta) \geq 0\}, \quad (14)$$

$$\mathcal{A}_3 = \{(\mathbf{E}, \theta) | e_1 - \beta(\theta) \leq 0, \nu(\theta)(e_1 - \beta(\theta)) + e_2 - \beta(\theta) \geq 0\}, \quad (15)$$

$$\mathcal{A}_4 = \{(\mathbf{E}, \theta) | \nu(\theta)(e_1 - \beta(\theta)) + e_2 - \beta(\theta) \leq 0, e_3 - \beta(\theta) + \gamma(\theta)(\text{tr} \mathbf{E} - 3\beta(\theta)) \geq 0\}. \quad (16)$$

By following a procedure similar to that used in Ref. (Lucchesi et al., 1995), it is possible to prove that for each $(\mathbf{E}, \theta) \in \text{Sym} \times [\theta_1, \theta_2]$, the unique solution $(\mathbf{T}, \mathbf{E}^a)$ to Eqs. (4)–(8) is given by

$$\mathbf{T} = \mathbb{C}(\theta)[\mathbf{E} - \beta(\theta) \mathbf{I}], \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{A}_1, \quad (17)$$

$$\mathbf{T} = \mathbf{0}, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_2, \tag{18}$$

$$\mathbf{T} = E(\theta)(e_1 - \beta(\theta))\mathbf{O}_1, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_3, \tag{19}$$

$$\mathbf{T} = \frac{E(\theta)}{1 - \nu^2(\theta)} \{ [e_1 - \beta(\theta) + \nu(\theta)(e_2 - \beta(\theta))] \mathbf{O}_1 + [e_2 - \beta(\theta) + \nu(\theta)(e_1 - \beta(\theta))] \mathbf{O}_2 \}, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_4, \tag{20}$$

where $\mathbf{0}$ is the null tensor;

$$\mathbf{E}^a = \mathbf{0}, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_1, \tag{21}$$

$$\mathbf{E}^a = \mathbf{E} - \beta(\theta)\mathbf{I}, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_2, \tag{22}$$

$$\mathbf{E}^a = [e_2 - \beta(\theta) + \nu(\theta)(e_1 - \beta(\theta))] \mathbf{O}_2 + [e_3 - \beta(\theta) + \nu(\theta)(e_1 - \beta(\theta))] \mathbf{O}_3, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_3, \tag{23}$$

$$\mathbf{E}^a = \frac{1}{1 - \nu(\theta)} [e_3 - \beta(\theta) + \nu(\theta)(e_1 + e_2 - e_3 - \beta(\theta))] \mathbf{O}_3, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_4. \tag{24}$$

It is easy to verify that in absence of temperature variations, the material characterised by the constitutive relations (4)–(8) conforms to the isothermal no-tension material studied by Del Piero (1989) and Panzeca and Polizzotto (1988).

The function $\hat{\mathbf{T}}$ that associates the stress given in Eqs. (17)–(20) to each $(\mathbf{E}, \theta) \in \text{Sym} \times [\theta_1, \theta_2]$ is continuous, isotropic, homogeneous of degree one and monotone with respect to \mathbf{E} ,

$$(\hat{\mathbf{T}}(\mathbf{E}_1, \theta) - \hat{\mathbf{T}}(\mathbf{E}_2, \theta)) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \geq 0, \quad \text{for } \mathbf{E}_1, \mathbf{E}_2 \in \text{Sym}. \tag{25}$$

Moreover, $\hat{\mathbf{T}}$ is differentiable with respect to \mathbf{E} in the interior of every region \mathcal{R}_i .

We observe that $\hat{\mathbf{T}}$ is non-injective. On the contrary, for $\theta \in [\theta_1, \theta_2]$ and $\mathbf{T} \in \text{Sym}^-$, the set constituted by all symmetric tensors \mathbf{A} such that $\hat{\mathbf{T}}(\mathbf{A}, \theta) = \mathbf{T}$ is the convex cone

$$\mathcal{C}(\mathbf{T}, \theta) = \left\{ \mathbf{A} \in \text{Sym} \mid \mathbf{A} = \mathbb{C}(\theta)^{-1}[\mathbf{T}] + \beta(\theta)\mathbf{I} + \hat{\mathbf{N}}(\mathbf{T}, \theta), \hat{\mathbf{N}}(\mathbf{T}, \theta) \in \mathcal{N}(\mathbf{T}) \right\}, \tag{26}$$

with $\mathcal{N}(\mathbf{T})$ defined in Eq. (11).

Let φ be a function defined from $\text{Sym} \times [\theta_1, \theta_2]$ with values in \mathbb{R} , depending on the elastic part of $\mathbf{E} - \beta(\theta)\mathbf{I}$,

$$\varphi(\mathbf{E}, \theta) = \tilde{\varphi}(\mathbb{C}(\theta)^{-1}[\hat{\mathbf{T}}(\mathbf{E}, \theta)] \theta); \tag{27}$$

then, we can express φ as function of \mathbf{T} and θ by setting

$$\tilde{\varphi}(\mathbf{T}, \theta) = \varphi(\hat{\mathbf{E}}(\mathbf{T}, \theta), \theta), \quad \text{with } \hat{\mathbf{E}}(\mathbf{T}, \theta) \in \mathcal{C}(\mathbf{T}, \theta). \tag{28}$$

2.2. The thermodynamical potentials

We refer to Truesdell and Noll (1965) and Carlson (1972) for a broad coverage of the thermodynamics and thermoelasticity concepts necessary for the theory developed here.

Let η be the *entropy* (per unit mass), ε the *internal energy*, $\psi = \varepsilon - \eta\theta$ the *free energy*, C_E the *specific heat at constant strain* and ρ the density. We assume that the following relations

$$\hat{\mathbf{T}}(\mathbf{E}, \theta) = \rho \partial_E \psi(\mathbf{E}, \theta), \quad (29)$$

$$\eta(\mathbf{E}, \theta) = -\partial_\theta \psi(\mathbf{E}, \theta), \quad (30)$$

$$\varepsilon(\mathbf{E}, \theta) = \psi(\mathbf{E}, \theta) + \theta \eta(\mathbf{E}, \theta), \quad (31)$$

$$C_E(\mathbf{E}, \theta) = \theta \partial_\theta \eta(\mathbf{E}, \theta) \quad (32)$$

hold. Moreover, for \mathbf{q} the *heat flux vector* per unit surface area and s the *heat supply* per unit mass, we have the energy equation

$$\rho \dot{\eta} \theta = -\operatorname{div} \mathbf{q} + s \rho, \quad (33)$$

where the dot denotes the time derivative.

Given the stress function $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}, \theta)$, from the thermostatic relation (29), by accounting for the symmetry of the fourth-order tensor $\partial_E \mathbf{T}(\mathbf{E}, \theta)$ and the equality $\partial_E \hat{\mathbf{T}}(\mathbf{E}, \theta)[\mathbf{E} - \beta(\theta)\mathbf{I}] = \hat{\mathbf{T}}(\mathbf{E}, \theta)$ both coming from the expressions (17)–(20), we obtain the free energy

$$\psi(\mathbf{E}, \theta) = \frac{1}{2\rho} (\mathbf{E} - \beta(\theta)\mathbf{I}) \cdot \hat{\mathbf{T}}(\mathbf{E}, \theta) + \zeta(\theta), \quad (34)$$

where $\zeta(\theta)$ is a material function which will be specified in the following. Note that due to Eqs. (4)–(6), the free energy depends solely on the elastic part \mathbf{E}^e of $\mathbf{E} - \beta(\theta)\mathbf{I}$, thereby satisfying Eq. (27). In view of Eqs. (17)–(20), from Eq. (34) we get

$$\psi(\mathbf{E}, \theta) = \zeta(\theta) + \frac{E(\theta)}{2(1+\nu(\theta))\rho} \left\{ \frac{\nu(\theta)}{1-2\nu(\theta)} (\operatorname{tr} \mathbf{E} - 3\beta(\theta))^2 + (e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + (e_3 - \beta(\theta))^2 \right\},$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_1$, (35)

$$\psi(\mathbf{E}, \theta) = \zeta(\theta), \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_2, \quad (36)$$

$$\psi(\mathbf{E}, \theta) = \zeta(\theta) + \frac{E(\theta)}{2\rho} (e_1 - \beta(\theta))^2, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{R}_3, \quad (37)$$

$$\psi(\mathbf{E}, \theta) = \zeta(\theta) + \frac{E(\theta)}{2(1-\nu^2(\theta))\rho} \left\{ (e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + 2\nu(\theta)(e_1 - \beta(\theta))(e_2 - \beta(\theta)) \right\}, \quad \text{for}$$

$$(\mathbf{E}, \theta) \in \mathcal{R}_4.$$

Let

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)} \quad \text{and} \quad 3\chi = \frac{E}{1 - 2\nu} \tag{39}$$

be the Lamé moduli and the coefficient of volumetric expansion of the material, respectively. From Eqs. (35)–(38), by using relations Eqs. (30) and (31) we can deduce the expressions for the entropy η and the internal energy ε .

For $(\mathbf{E}, \theta) \in \mathcal{R}_1$,

$$\begin{aligned} \eta(\mathbf{E}, \theta) = & -\xi'(\theta) - \frac{\lambda'(\theta)}{2\rho} (\text{tr}\mathbf{E} - 3\beta(\theta))^2 - \frac{\mu'(\theta)}{\rho} \left\{ (e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + (e_3 - \beta(\theta))^2 \right\} \\ & + \frac{3\chi(\theta)\beta'(\theta)}{\rho} (\text{tr}\mathbf{E} - 3\beta(\theta)), \end{aligned} \tag{40}$$

$$\begin{aligned} \varepsilon(\mathbf{E}, \theta) = & \zeta(\theta) - \theta\xi'(\theta) + \frac{\lambda(\theta) - \theta\lambda'(\theta)}{2\rho} (\text{tr}\mathbf{E} - 3\beta(\theta))^2 + \frac{\mu(\theta) - \theta\mu'(\theta)}{\rho} \left\{ (e_1 - \beta(\theta))^2 \right. \\ & \left. + (e_2 - \beta(\theta))^2 + (e_3 - \beta(\theta))^2 \right\} + \frac{3\theta\chi(\theta)\beta'(\theta)}{\rho} (\text{tr}\mathbf{E} - 3\beta(\theta)); \end{aligned} \tag{41}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_2$,

$$\eta(\mathbf{E}, \theta) = -\xi'(\theta), \tag{42}$$

$$\varepsilon(\mathbf{E}, \theta) = \zeta(\theta) - \theta\xi'(\theta); \tag{43}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_3$,

$$\eta(\mathbf{E}, \theta) = -\xi'(\theta) - \frac{E'(\theta)}{2\rho} (e_1 - \beta(\theta))^2 + \frac{E(\theta)\beta'(\theta)}{\rho} (e_1 - \beta(\theta)), \tag{44}$$

$$\varepsilon(\mathbf{E}, \theta) = \zeta(\theta) - \theta\xi'(\theta) + \frac{E(\theta) - \theta E'(\theta)}{2\rho} (e_1 - \beta(\theta))^2 + \frac{\theta E(\theta)\beta'(\theta)}{\rho} (e_1 - \beta(\theta)); \tag{45}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_4$,

$$\begin{aligned} \eta(\mathbf{E}, \theta) = & -\xi'(\theta) - \frac{E'(\theta)(1 - \nu(\theta)^2) + 2E(\theta)\nu(\theta)\nu'(\theta)}{2\rho(1 - \nu(\theta)^2)^2} \left\{ (e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 \right\} \\ & - \frac{E'(\theta)\nu(\theta)(1 - \nu(\theta)^2) + E(\theta)\nu'(\theta)(1 + \nu(\theta)^2)}{\rho(1 - \nu(\theta)^2)^2} (e_1 - \beta(\theta))(e_2 - \beta(\theta)) \\ & + \frac{E(\theta)\beta'(\theta)}{\rho(1 - \nu(\theta))} (e_1 + e_2 - 2\beta(\theta)), \end{aligned} \tag{46}$$

$$\begin{aligned} \varepsilon(\mathbf{E}, \theta) = & \zeta(\theta) - \theta \zeta'(\theta) + \frac{(E(\theta) - \theta E'(\theta))(1 - \nu(\theta)^2) - 2\theta E(\theta)\nu(\theta)\nu'(\theta)}{2\rho(1 - \nu(\theta)^2)^2} \{ (e_1 - \beta(\theta))^2 + (e_2 \\ & - \beta(\theta))^2 \} + \frac{\theta E(\theta)\beta'(\theta)}{\rho(1 - \nu(\theta))} (e_1 + e_2 - 2\beta(\theta)) \\ & + \frac{(E(\theta) - \theta E'(\theta))\nu(\theta)(1 - \nu(\theta)^2) - \theta E(\theta)\nu'(\theta)(1 + \nu(\theta)^2)}{\rho(1 - \nu(\theta)^2)^2} (e_1 - \beta(\theta))(e_2 - \beta(\theta)); \end{aligned} \quad (47)$$

where ' denotes the derivative with respect to θ .

From Eqs. (17)–(20), (40), (42), (44) and (46), we obtain the *Maxwell relation*

$$\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta) = -\rho \partial_E \eta(\mathbf{E}, \theta). \quad (48)$$

It is easily verified that the internal energy and entropy both satisfy condition (27). In particular, in view of Eq. (28), the internal energy can be expressed as a function of \mathbf{T} and θ , $\varepsilon = \bar{\varepsilon}(\mathbf{T}, \theta)$. Thus, we can define the *enthalpy* per unit mass

$$\mathcal{H}(\mathbf{T}, \theta) = \bar{\varepsilon}(\mathbf{T}, \theta) - \frac{1}{\rho} \mathbf{T} \cdot \hat{\mathbf{E}}(\mathbf{T}, \theta), \quad \hat{\mathbf{E}}(\mathbf{T}, \theta) \in \mathcal{C}(\mathbf{T}, \theta). \quad (49)$$

And in view of the definition of $\mathcal{C}(\mathbf{T}, \theta)$, we have

$$\mathcal{H}(\mathbf{T}, \theta) = \bar{\varepsilon}(\mathbf{T}, \theta) - \frac{1}{\rho} \left(\mathbf{T} \cdot \mathbb{C}(\theta)^{-1} [\mathbf{T}] + \beta(\theta) \text{tr} \mathbf{T} \right). \quad (50)$$

2.3. The specific heat at constant strain

In this subsection we omit to indicate the dependence of χ , β , λ , μ , E and ν on θ .

From Eq. (32), by accounting for Eqs. (40), (42), (44) and (46), we get the specific heat C_E at constant strain which has different expressions in the four regions.

For $(\mathbf{E}, \theta) \in \mathcal{R}_1$, by virtue of Eq. (40), we have

$$C_{E1}(\mathbf{E}, \theta) = -\theta \zeta''(\theta) + \theta \left\{ \frac{6\chi'\beta' + 3\chi\beta''}{\rho} (\text{tr} \mathbf{E} - 3\beta) - \frac{\lambda''}{2\rho} (\text{tr} \mathbf{E} - 3\beta)^2 - \frac{\mu''}{\rho} \|\mathbf{E} - \beta \mathbf{I}\|^2 - \frac{9\chi\beta'^2}{\rho} \right\}, \quad (51)$$

from which we obtain

$$\begin{aligned} \zeta''(\theta) = & -\frac{C_{E1}(\mathbf{E}, \theta)}{\theta} - \frac{\lambda''}{2\rho} (\text{tr} \mathbf{E})^2 - \frac{\mu''}{\rho} \|\mathbf{E}\|^2 + \frac{6\chi'\beta' + 3\chi\beta'' + 3\chi''\beta}{\rho} \text{tr} \mathbf{E} - \frac{9}{\rho} \left\{ \frac{\chi''\beta^2}{2} + 2\chi'\beta\beta' \right. \\ & \left. + \chi\beta\beta'' + \chi\beta'^2 \right\}. \end{aligned} \quad (52)$$

Since ζ'' depends solely on temperature, the quantity $C_{E1}(\mathbf{E}, \theta) - \frac{\theta}{\rho} \left\{ -\frac{\lambda''}{2} (\text{tr} \mathbf{E})^2 - \mu'' \|\mathbf{E}\|^2 + (6\chi'\beta' + 3\chi\beta'' + 3\chi''\beta) \text{tr} \mathbf{E} \right\}$ must be a function of θ , say $z_1(\theta)$, and thus

$$C_{E1}(\mathbf{E}, \theta) = z_1(\theta) + (\theta/\rho) \left\{ -(\lambda''/2) (\text{tr} \mathbf{E})^2 - \mu'' \|\mathbf{E}\|^2 + (6\chi'\beta' + 3\chi\beta'' + 3\chi''\beta) \text{tr} \mathbf{E} \right\}. \quad (53)$$

Then from Eq. (52), by accounting for Eq. (53), we obtain

$$\xi''(\theta) = -\frac{z_1(\theta)}{\theta} - \frac{9}{\rho} \left\{ \frac{\chi''\beta^2}{2} + 2\chi'\beta\beta' + \chi\beta\beta'' + \chi\beta'^2 \right\}. \tag{54}$$

Since the thermodynamic potentials are defined within an arbitrary constant, we assume that they vanish for $\mathbf{E} = \mathbf{0}$ and $\theta = \theta_0$; in other words, we suppose that the equalities

$$\xi(\theta_0) = \xi'(\theta_0) = 0 \tag{55}$$

hold. From Eq. (54), in view of Eq. (55), we deduce the following relation

$$\xi(\theta) = \int_{\theta_0}^{\theta} \left(1 - \frac{\theta}{\theta'} \right) z_1(\theta') d\theta' - \frac{9\chi\beta^2}{2\rho}, \tag{56}$$

which allows determination of the function $\xi(\theta)$, once the specific heat in the region \mathcal{R}_1 , and then function $z_1(\theta)$ are known.

For $(\mathbf{E}, \theta) \in \mathcal{R}_2$, Eq. (42) yields

$$C_{E2}(\mathbf{E}, \theta) = -\theta\xi''(\theta), \tag{57}$$

and then C_{E2} depends on temperature alone.

For $(\mathbf{E}, \theta) \in \mathcal{R}_3$, by virtue of Eq. (44) we have

$$C_{E3}(\mathbf{E}, \theta) = -\theta\xi''(\theta) + \theta \left\{ \frac{2E'\beta' + E\beta''}{\rho}(e_1 - \beta) - \frac{E''}{2\rho}(e_1 - \beta)^2 - \frac{E\beta'^2}{\rho} \right\}, \tag{58}$$

from which we obtain

$$\begin{aligned} \xi''(\theta) = & -\frac{C_{E3}(\mathbf{E}, \theta)}{\theta} + \frac{1}{\rho} \left\{ -\frac{E''}{2}e_1^2 + (E''\beta + 2E'\beta' + E\beta'')e_1 \right\} \\ & - \frac{1}{\rho} \left\{ \frac{E''\beta^2}{2} + 2E'\beta\beta' + E\beta\beta'' + E\beta'^2 \right\}. \end{aligned} \tag{59}$$

We can put $z_3(\theta) = C_{E3}(\mathbf{E}, -\theta)(\theta/\rho)\{- (E''/2)e_1^2 + (E''\beta + 2E'\beta' + E\beta'')e_1\}/\theta$, and thus obtain

$$C_{E3}(\mathbf{E}, \theta) = z_3(\theta) + \frac{\theta}{\rho} \left\{ -\frac{E''}{2}e_1^2 + (E''\beta + 2E'\beta' + E\beta'')e_1 \right\}. \tag{60}$$

By accounting for Eq. (60), (59) yields

$$z_3(\theta) = -\theta\xi''(\theta) - \frac{\theta}{\rho} \left\{ \frac{E''\beta^2}{2} + 2E'\beta\beta' + E\beta\beta'' + E\beta'^2 \right\}. \tag{61}$$

Finally, for $(\mathbf{E}, \theta) \in \mathcal{R}_4$, by accounting for Eq. (46), we get

$$\begin{aligned} C_{E4}(\mathbf{E}, \theta) = & -\theta\xi''(\theta) + \theta \left\{ -\zeta'_1[(e_1 - \beta)^2 + (e_2 - \beta)^2] - \zeta'_2(e_1 - \beta)(e_2 - \beta) + (2\beta'\zeta_1 + \beta'\zeta_2 + \zeta'_3) \right. \\ & \left. \times (e_1 + e_2 - 2\beta) - 2\beta'\zeta_3 \right\}, \end{aligned} \tag{62}$$

where

$$\zeta_1 = \frac{E'(1-\nu^2) + 2Ev\nu'}{2\rho(1-\nu^2)^2}, \quad (63)$$

$$\zeta_2 = \frac{E'\nu(1-\nu^2) + Ev'(1+\nu^2)}{\rho(1-\nu^2)^2}, \quad (64)$$

$$\zeta_3 = \frac{E\beta'}{\rho(1-\nu)}. \quad (65)$$

From Eq. (62), we obtain

$$\begin{aligned} \xi''(\theta) = & -\frac{C_{E4}(\mathbf{E}, \theta)}{\theta} - \zeta_1'(e_1^2 + e_2^2) - \zeta_2'e_1e_2 + (2\beta\zeta_1' + 2\beta'\zeta_1 + \beta\zeta_2' + \beta'\zeta_2 + \zeta_3')(e_1 + e_2) - 2\beta^2\zeta_1' \\ & - 4\beta\beta'\zeta_1 - \beta^2\zeta_2' - 2\beta\beta'\zeta_2 - 2\beta\zeta_3' - 2\beta'\zeta_3. \end{aligned} \quad (66)$$

Since ξ'' depends solely on temperature, $C_{E4}(\mathbf{E}, \theta) - \theta\{-\zeta_1'(e_1^2 + e_2^2) - \zeta_2'e_1e_2 + (2\beta\zeta_1' + 2\beta'\zeta_1 + \beta\zeta_2' + \beta'\zeta_2 + \zeta_3')(e_1 + e_2)\}$ must be a function of θ , say $z_4(\theta)$, and thus

$$C_{E4}(\mathbf{E}, \theta) = z_4(\theta) + \theta\left\{-\zeta_1'(e_1^2 + e_2^2) - \zeta_2'e_1e_2 + (2\beta\zeta_1' + 2\beta'\zeta_1 + \beta\zeta_2' + \beta'\zeta_2 + \zeta_3')(e_1 + e_2)\right\}. \quad (67)$$

From Eq. (66), by accounting for Eq. (67) we obtain

$$z_4(\theta) = -\theta\xi''(\theta) - \theta\left(2\beta^2\zeta_1' + 4\beta\beta'\zeta_1 + \beta^2\zeta_2' + 2\beta\beta'\zeta_2 + 2\beta\zeta_3' + 2\beta'\zeta_3\right). \quad (68)$$

Since function $\xi(\theta)$ and, therefore, $\xi''(\theta)$ are the same in the four regions, by comparing Eq. (57) and Eqs. (52), (59) and (66), relationships between the specific heat in \mathcal{R}_2 and in the remaining regions follow,

$$\begin{aligned} C_{E1}(\mathbf{E}, \theta) = & C_{E2}(\theta) - \frac{9\theta}{\rho}\left\{\frac{\chi''\beta^2}{2} + 2\chi'\beta\beta' + \chi\beta\beta'' + \chi\beta'^2\right\} \\ & + \frac{\theta}{\rho}\left\{-\frac{\lambda''}{2}(\text{tr}\mathbf{E})^2 - \mu''\|\mathbf{E}\|^2 + (6\chi'\beta' + 3\chi\beta'' + 3\chi''\beta)\text{tr}\mathbf{E}\right\}, \end{aligned} \quad (69)$$

$$\begin{aligned} C_{E3}(\mathbf{E}, \theta) = & C_{E2}(\theta) - \frac{\theta}{\rho}\left\{\frac{E''\beta^2}{2} + 2E'\beta\beta' + E\beta\beta'' + E\beta'^2\right\} \\ & + \frac{\theta}{\rho}\left\{-\frac{E''}{2}e_1^2 + (E''\beta + 2E'\beta' + E\beta'')e_1\right\}, \end{aligned} \quad (70)$$

$$\begin{aligned} C_{E4}(\mathbf{E}, \theta) = & C_{E2}(\theta) + \theta\left\{-\zeta_1'(e_1^2 + e_2^2) - \zeta_2'e_1e_2 + (2\beta\zeta_1' + 2\beta'\zeta_1 + \beta\zeta_2' + \beta'\zeta_2 + \zeta_3')(e_1 + e_2)\right. \\ & \left. - 2\beta^2\zeta_1' - 4\beta\beta'\zeta_1 - \beta^2\zeta_2' - 2\beta\beta'\zeta_2 - 2\beta\zeta_3' - 2\beta'\zeta_3\right\}. \end{aligned} \quad (71)$$

From Eqs. (69)–(71), it follows that if the elastic constants are independent of temperature and moreover, $\beta'' = 0$ (namely the thermal expansion is a linear function of θ), then the specific heats C_{E1} , C_{E3} and C_{E4} do not depend on strain \mathbf{E} , but on temperature alone,

$$C_{E1}(\mathbf{E}, \theta) = C_{E2}(\theta) - \frac{9\theta}{\rho} \chi \beta'^2, \tag{72}$$

$$C_{E3}(\mathbf{E}, \theta) = C_{E2}(\theta) - \frac{\theta}{\rho} E \beta'^2, \tag{73}$$

$$C_{E4}(\mathbf{E}, \theta) = C_{E2}(\theta) - \frac{2\theta}{\rho(1-\nu)} E \beta'^2. \tag{74}$$

On the other hand, if $\beta'' \neq 0$, C_{E1} is a linear function of \mathbf{E} ,

$$C_{E1}(\mathbf{E}, \theta) = C_{E2}(\theta) - \frac{9\theta}{\rho} (\chi \beta \beta'' + \chi \beta'^2) + 3 \frac{\theta}{\rho} \chi \beta'' \text{tr} \mathbf{E}; \tag{75}$$

while on the contrary, C_{E3} and C_{E4} are non-linear functions of \mathbf{E} , namely,

$$C_{E3}(\mathbf{E}, \theta) = C_{E2}(\theta) - \frac{E\theta}{\rho} (\beta \beta'' + \beta'^2) + \frac{\theta}{\rho} E \beta'' e_1, \tag{76}$$

$$C_{E4}(\mathbf{E}, \theta) = C_{E2}(\theta) - \frac{E\theta}{\rho(1-\nu)} (\beta \beta'' + \beta'^2) + \frac{E\beta''\theta}{\rho(1-\nu)} (e_1 + e_2). \tag{77}$$

2.4. The specific heat at constant stress

We shall denote the *specific heat at constant stress* per unit mass by

$$C_T(\mathbf{T}, \theta) = \partial_\theta \ell(\mathbf{T}, \theta), \tag{78}$$

with ℓ defined in Eq. (49). Setting

$$\bar{\eta}(\mathbf{T}, \theta) = \eta(\hat{\mathbf{E}}(\mathbf{T}, \theta), \theta), \tag{79}$$

we can prove, by analogy to Eq. (32), that

$$C_T(\mathbf{T}, \theta) = \theta \partial_\theta \bar{\eta}(\mathbf{T}, \theta). \tag{80}$$

In fact, from Eqs. (78) and (49), by virtue of Eqs. (29)–(31), we get

$$\begin{aligned} C_T(\mathbf{T}, \theta) &= \partial_\theta \varepsilon(\mathbf{E}, \theta) + \partial_E \varepsilon(\mathbf{E}, \theta) \cdot \partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) - \frac{1}{\rho} \mathbf{T} \cdot \partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) \\ &= \partial_\theta \psi(\mathbf{E}, \theta) + \eta(\mathbf{E}, \theta) + \theta \partial_\theta \eta(\mathbf{E}, \theta) + (\partial_E \psi(\mathbf{E}, \theta) + \theta \partial_E \eta(\mathbf{E}, \theta)) \cdot \partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) \\ &\quad - \frac{1}{\rho} \mathbf{T} \cdot \partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) = \theta \partial_\theta \eta(\mathbf{E}, \theta) + \theta \partial_E \eta(\mathbf{E}, \theta) \cdot \partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) = \theta \partial_\theta \bar{\eta}(\mathbf{T}, \theta). \end{aligned} \tag{81}$$

The derivative $\partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta)$ in Eq. (81) is well defined for each possible choice of $\hat{\mathbf{E}}(\mathbf{T}, \theta)$ in $\mathcal{C}(\mathbf{T}, \theta)$ and, in

view of Eq. (26), it holds that

$$\partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) = (\mathbb{C}(\theta)^{-1})' [\mathbf{T}] + \beta'(\theta) \mathbf{I} + \partial_\theta \hat{\mathbf{N}}(\mathbf{T}, \theta), \quad (82)$$

with

$$\mathbb{C}(\theta)^{-1} = \frac{1 + \nu(\theta)}{E(\theta)} \mathbb{I} - \frac{\nu(\theta)}{E(\theta)} \mathbf{I} \otimes \mathbf{I}. \quad (83)$$

Now, we intend to establish a relationship between C_E and C_T . Namely, we shall prove that

$$C_T(\mathbf{T}, \theta) - C_E(\mathbf{E}, \theta) = \frac{\theta}{\rho} \partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta) \cdot \mathbb{C}(\theta)^{-1} [\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta)]. \quad (84)$$

Note that due to Eq. (83), the right-hand side of Eq. (84) is equal to $(\theta/\rho)\{[(1 + \nu(\theta))/E(\theta)]\|\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta)\|^2 - (\nu(\theta)/E(\theta))(\text{tr} \partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta))^2\}$, and then, it is of the order $O(\delta^2)$.

In order to prove Eq. (84), we observe that, in view of Eqs. (80), (32) and (48), we can write,

$$\begin{aligned} C_T(\mathbf{T}, \theta) - C_E(\mathbf{E}, \theta) &= \theta \{ \partial_\theta \eta(\hat{\mathbf{E}}(\mathbf{T}, \theta), \theta) - \partial_\theta \eta(\mathbf{E}, \theta) \} = \theta \partial_{E\eta}(\mathbf{E}, \theta) \cdot \partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) \\ &= -\frac{\theta}{\rho} \partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta) \cdot \partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta). \end{aligned} \quad (85)$$

On the other hand, using the chain rule we get

$$\partial_\theta \hat{\mathbf{E}}(\mathbf{T}, \theta) = -\partial_T \hat{\mathbf{E}}(\mathbf{T}, \theta) [\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta)] + \partial_\theta \hat{\mathbf{N}}(\mathbf{T}, \theta) \quad (86)$$

and finally, accounting for the orthogonality of $\partial_\theta \hat{\mathbf{N}}(\mathbf{T}, \theta)$ and $\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta)$ due to Eq. (12), Eq. (85) becomes

$$C_T(\mathbf{T}, \theta) - C_E(\mathbf{E}, \theta) = \frac{\theta}{\rho} \partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta) \cdot \partial_T \hat{\mathbf{E}}(\mathbf{T}, \theta) [\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta)]. \quad (87)$$

From Eq. (26), we obtain the relation

$$\partial_T \hat{\mathbf{E}}(\mathbf{T}, \theta) = \mathbb{C}(\theta)^{-1} + \partial_T \hat{\mathbf{N}}(\mathbf{T}, \theta); \quad (88)$$

moreover, it is an easy matter to prove that in light of Eq. (12), it holds that $\partial_T \hat{\mathbf{N}}(\mathbf{T}, \theta) [\mathbf{O}_i] = 0$, for every i such that the i th eigenvalue of \mathbf{T} is negative. From Eqs. (17)–(20) we deduce that $\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta)$ has the same eigenvectors as $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}, \theta)$ and can thus conclude that

$$\partial_T \hat{\mathbf{N}}(\mathbf{T}, \theta) [\partial_\theta \hat{\mathbf{T}}(\mathbf{E}, \theta)] = \mathbf{0}. \quad (89)$$

Finally, by accounting for Eqs. (88) and (89), relation (84) follows from Eq. (87).

2.5. The energy equation

In order to complete the system of constitutive equations, we assume the usual relation for heat flux

$$\mathbf{q} = -\kappa(\theta) \mathbf{g}, \quad (90)$$

where $\kappa(\theta) \geq 0$ is the *conductivity coefficient*, and $\mathbf{g} = \text{grad} \theta$ the temperature gradient.

The material having constitutive relations (35)–(38), (40)–(47), (52), (17)–(20), (51), (57), (58), (62) and (90) is wholly characterised by five temperature-dependent functions $E(\theta)$, $\nu(\theta)$, $\beta(\theta)$, $\kappa(\theta)$ and $\xi(\theta)$. Moreover, from relations (48) and (32) we can deduce that

$$\dot{\eta}(\mathbf{E}, \theta) = \partial_E \eta(\mathbf{E}, \theta) \cdot \dot{\mathbf{E}} + \partial_\theta \eta(\mathbf{E}, \theta) \dot{\theta} = -\frac{1}{\rho} \partial_\theta \mathbf{T}(\mathbf{E}, \theta) \cdot \dot{\mathbf{E}} + \frac{1}{\theta} C_E(\mathbf{E}, \theta) \dot{\theta}. \tag{91}$$

By then accounting for Eqs. (17)–(20) and (33), we can write the energy equation in the four regions \mathcal{R}_i .

For $(\mathbf{E}, \theta) \in \mathcal{R}_1$,

$$-\text{div} \mathbf{q} + \theta \{ 2\mu'(\theta) \mathbf{E} \cdot \dot{\mathbf{E}} + \lambda'(\theta) \text{tr} \mathbf{E} \text{tr} \dot{\mathbf{E}} - 3(\chi\beta)' \text{tr} \dot{\mathbf{E}} \} + \rho s = \rho C_{E1}(\mathbf{E}, \theta) \dot{\theta}, \tag{92}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_2$,

$$-\text{div} \mathbf{q} + \rho s = -\rho \theta \xi''(\theta) \dot{\theta}, \tag{93}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_3$,

$$-\text{div} \mathbf{q} + \theta \left\{ E'(\theta) (e_1 - \beta(\theta)) \mathbf{O}_1 \cdot \dot{\mathbf{E}} - E(\theta) \beta'(\theta) \mathbf{O}_1 \cdot \dot{\mathbf{E}} \right\} + \rho s = -\rho C_{E3}(\mathbf{E}, \theta) \dot{\theta}, \tag{94}$$

for $(\mathbf{E}, \theta) \in \mathcal{R}_4$,

$$\begin{aligned} -\text{div} \mathbf{q} + \theta \left\{ \frac{E'(\theta)(1 - \nu(\theta)^2) + 2E(\theta)\nu(\theta)\nu'(\theta)}{(1 - \nu(\theta)^2)^2} [(e_1 + \nu(\theta)e_2(1 + \nu(\theta))\beta(\theta)) \mathbf{O}_1 \cdot \dot{\mathbf{E}} + (e_2 + \nu(\theta)e_1 \right. \\ \left. - (1 + \nu(\theta))\beta(\theta)) \mathbf{O}_2 \cdot \dot{\mathbf{E}}] + \frac{E(\theta)}{1 - \nu(\theta)^2} [\nu'(\theta)e_2 - \beta'(\theta) - \nu'(\theta)\beta(\theta) - \nu(\theta)\beta'(\theta)] \mathbf{O}_1 \cdot \dot{\mathbf{E}} \right. \\ \left. + (\nu'(\theta)e_1 - \beta'(\theta) - \nu'(\theta)\beta(\theta) - \nu(\theta)\beta'(\theta)) \mathbf{O}_2 \cdot \dot{\mathbf{E}} \right\} + \rho s \\ = \rho C_{E4}(\mathbf{E}, \theta) \dot{\theta}. \end{aligned} \tag{95}$$

Thus, the basic equations of the thermoelastic theory are the strain-displacement relation, the equilibrium equation, constitutive relations (17)–(20) and (90) for stress and heat flux, respectively, and finally, the equilibrium energy equations (92)–(95). The system of equations obtained is coupled because the temperature coefficient on the left-hand side and the coefficient of the derivative of temperature on the right-hand side of the energy equations (92)–(95) depend on the strain and strain rate. In particular, if we assume separately

$$\mathbf{E} = O(\delta), \quad \beta(\theta) = O(\delta), \quad \beta'(\theta) = O(\delta), \quad \dot{\mathbf{E}} = O(\delta), \quad \dot{\theta} = O(\delta), \tag{96}$$

energy equations (92)–(95) can be simplified; in fact, by disregarding terms of order $O(\delta^2)$, we obtain

$$-\text{div} \mathbf{q} + \rho s = -\rho \theta \xi''(\theta) \dot{\theta} \tag{97}$$

for all regions, and thermo-mechanical uncoupling occurs.

Finally, if for temperature the further condition

$$\theta = \theta_0 + O(\delta), \tag{98}$$

holds, with θ_0 as the reference temperature, then we have

$$\beta(\theta) = \beta(\theta_0) + \beta'(\theta_0)(\theta - \theta_0) + o(\delta), \quad (99)$$

(given a mapping B from a neighbourhood of 0 in \mathbb{R} into a vector space with norm $\|\cdot\|$, we write $B(\delta) = o(\delta)$ if for each $k > 0$ there is $k' > 0$ such that $\|B(\delta)\| < k|\delta|$, whenever $|\delta| < k'$) and thus, taking into account that $\beta(\theta_0) = 0$, we get within an error of order $o(\delta)$

$$\beta(\theta) = \beta'(\theta_0)(\theta - \theta_0) \quad (100)$$

in a neighbourhood of $\theta = \theta_0$. The quantity

$$\alpha = \beta'(\theta_0) \quad (101)$$

is the *linear coefficient of thermal expansion*. Moreover, in this case, functions E , ν , β , κ and C_E must be supposed temperature-independent and coincident with their value at θ_0 (Carlson, 1972).

The equilibrium problem for no-tension solids has been studied in recent years and the existence of a solution has been proved only for a rather restricted class of load conditions (Anzellotti, 1985; Giaquinta and Giusti, 1985). However, the uniqueness of the solution is guaranteed in terms of stress alone, in the sense that different displacement and strain fields can correspond to the same stress field. Similar considerations can be made for a no-tension material with constitutive equations (4)–(8).

3. An example

In this section, we compare the thermodynamical behaviour of a linear elastic and a no-tension elastic body in order to reveal eventual differences. Specifically, under the hypothesis of thermo-mechanical uncoupling, we consider two spherical containers \mathcal{S}^e and \mathcal{S}^m with inner radius r_1 and outer radius r_2 , subjected to two uniform radial pressures p_1 and p_2 acting on the inner and outer boundaries, respectively (Fig. 1).

Container \mathcal{S}^e is made of a linear elastic material, and \mathcal{S}^m is made of a no-tension elastic material with constitutive equations (4)–(8). In the initial state, the two containers are subjected to a steady temperature distribution. The temperatures assigned to the containers boundaries as well as the radial pressures are chosen in such a way that the initial state \mathcal{F}^m of \mathcal{S}^m coincides with the initial state \mathcal{F}^e of \mathcal{S}^e . In particular, \mathcal{S}^e and \mathcal{S}^m are characterised by the same purely compressive stress field, and fractures are, therefore, absent in \mathcal{S}^m . By means of an adiabatic process, with the boundary radial pressures kept unchanged, \mathcal{S}^e reaches the final state \mathcal{F}^e , characterised by an equilibrium temperature $\bar{\theta}_e$ and a stress field exhibiting tractions. The unknown temperature $\bar{\theta}_e$ is calculated explicitly by using the first principle of thermodynamics, imposing that the variation in internal energy equals the work done by the external loads passing from the state \mathcal{D}^e to \mathcal{F}^e . We proceed in an analogous manner for container \mathcal{S}^m , which adiabatically reaches the final state \mathcal{F}^m corresponding to the equilibrium temperature $\bar{\theta}_m$. In this case, due to our choice of the ratio p_2/p_1 in the interval $[r_1^2/r_2^2, (2r_1^3 + r_2^3)/3r_2^3]$ (Bennati et al., 1997), the container presents cracking in the region $r_1 \leq r \leq r_0$ with r_0 , less than or equal to r_2 , depending on r_1 , r_2 and p_2/p_1 and is entirely compressed in the remaining region (Fig. 2).

Finally, the variation in entropy passing from the initial to the final state is calculated for both \mathcal{S}^m and \mathcal{S}^e , and then, the work lost in the two cases is evaluated.

At the beginning of the process, containers \mathcal{S}^e and \mathcal{S}^m are subjected to a steady temperature distribution θ depending on the radius r ,

$$\theta(r) - \theta_0 = \frac{r_1 r_2 (\vartheta_1 - \vartheta_2) 1}{r_2 - r_1} \frac{1}{r} + \frac{r_2 \vartheta_2 - r_1 \vartheta_1}{r_2 - r_1}, \quad (102)$$

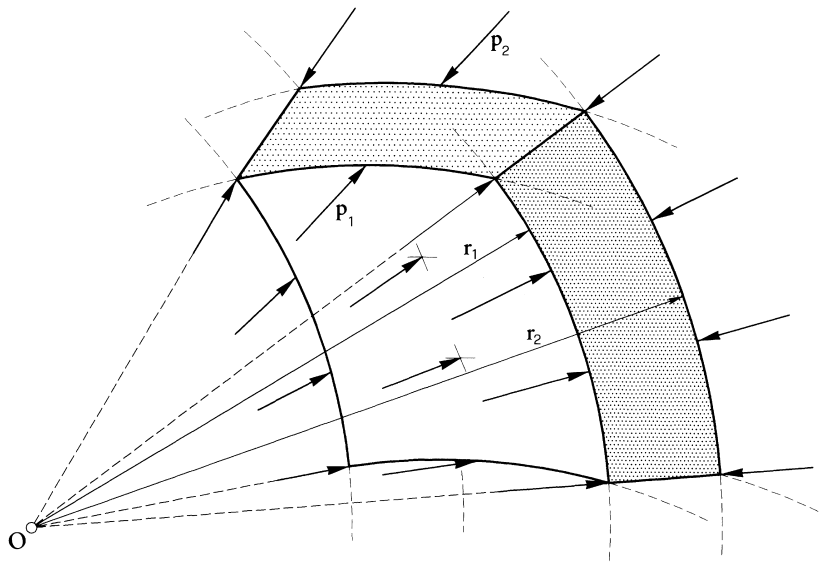


Fig. 1. The spherical container.

with θ_0 as the reference temperature; $\theta_1 = \vartheta_1 + \theta_0$ and $\theta_2 = \vartheta_2 + \theta_0$ are the temperature of the inner and outer boundaries, respectively.

We assume that Young's modulus E is independent of temperature and Poisson's ratio ν is zero. Moreover, for thermal expansion β we assume the linear temperature function $\beta(\theta) = \alpha(\theta - \theta_0)$.

We start by determining the stress field corresponding to a linear elastic material. Denoting by σ_r and σ_t the radial and tangential components of the stress, respectively, we have

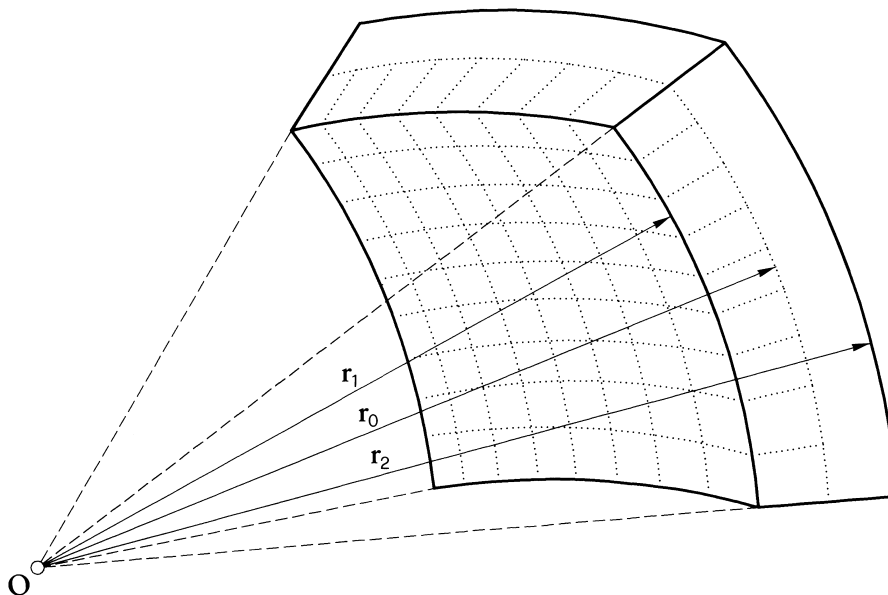


Fig. 2. The cracked spherical container.

$$\sigma_r(r) = \frac{r_1^3 r_2^3}{r_2^3 - r_1^3} [\alpha E (\vartheta_1 - \vartheta_2) - p_1 + p_2] \frac{1}{r^3} - \frac{r_1 r_2}{r_2 - r_1} \alpha E (\vartheta_1 - \vartheta_2) \frac{1}{r} + \frac{1}{r_2^3 - r_1^3} [p_1 r_1^3 - p_2 r_2^3 + \alpha E r_1 r_2 (r_1 + r_2) (\vartheta_1 - \vartheta_2)], \quad (103)$$

$$\sigma_t(r) = -\frac{r_1^3 r_2^3}{2(r_2^3 - r_1^3)} [\alpha E (\vartheta_1 - \vartheta_2) - p_1 + p_2] \frac{1}{r^3} - \frac{r_1 r_2}{2(r_2 - r_1)} \alpha E (\vartheta_1 - \vartheta_2) \frac{1}{r} + \frac{1}{r_2^3 - r_1^3} [p_1 r_1^3 - p_2 r_2^3 + \alpha E r_1 r_2 (r_1 + r_2) (\vartheta_1 - \vartheta_2)]. \quad (104)$$

It is easy matter to prove that for fixed r_1 , r_2 , E and α , there exist values of parameters θ_1 , θ_2 , p_1 and p_2 such that the stress fields (103) and (104) is negative semi-definite; and thus, it is the solution corresponding to both a linear and a no-tension elastic material. The radial component u of the displacement field associated with the stress (103) and (104) is

$$u(r) = \frac{r_1^3 r_2^3}{2(r_2^3 - r_1^3)} \left[\frac{p_1 - p_2}{E} - \alpha (\vartheta_1 - \vartheta_2) \right] \frac{1}{r^2} + \frac{r_1 r_2}{2(r_2 - r_1)} \alpha (\vartheta_1 - \vartheta_2) + \left[\frac{p_1 r_1^3 - p_2 r_2^3}{E(r_2^3 - r_1^3)} + \alpha \frac{\vartheta_2 r_2 - \vartheta_1 r_1}{r_2 - r_1} + \alpha \frac{r_1 r_2 (r_1 + r_2)}{r_2^3 - r_1^3} (\vartheta_1 - \vartheta_2) \right] r. \quad (105)$$

In particular, the displacements of the points belonging to the inner and outer boundaries, corresponding to the initial state are

$$u_{i1} = u(r_1) = r_1 \frac{p_1 r_1^3 - p_2 r_2^3}{E(r_2^3 - r_1^3)} + \frac{r_1 r_2^3 (p_1 - p_2)}{2E(r_2^3 - r_1^3)} + \alpha r_1 \frac{\vartheta_2 r_2 - \vartheta_1 r_1}{r_2 - r_1} + 3\alpha \frac{r_1^2 r_2 (r_1 + r_2)}{2(r_2^3 - r_1^3)} (\vartheta_1 - \vartheta_2) \quad (106)$$

and

$$u_{i2} = u(r_2) = r_2 \frac{p_1 r_1^3 - p_2 r_2^3}{E(r_2^3 - r_1^3)} + \frac{r_1^3 r_2 (p_1 - p_2)}{2E(r_2^3 - r_1^3)} + \alpha r_2 \frac{\vartheta_2 r_2 - \vartheta_1 r_1}{r_2 - r_1} + 3\alpha \frac{r_1 r_2^2 (r_1 + r_2)}{2(r_2^3 - r_1^3)} (\vartheta_1 - \vartheta_2), \quad (107)$$

respectively.

By assuming that the specific heat at constant strain C_E is constant, in view of Eqs. (56) and (53), we obtain ξ as a function of temperature θ

$$\xi(\theta) = C_E (\theta - \theta_0) - C_E \theta \ln \left(\frac{\theta}{\theta_0} \right) - \frac{3E\alpha^2}{2\rho} (\theta - \theta_0)^2. \quad (108)$$

From Eq. (41), by accounting for Eq. (108), we get the internal energy per unit mass ε_i as a function of r , corresponding to the initial state

$$\varepsilon_i(r) = C_E (\theta(r) - \theta_0) + \frac{3E\alpha^2}{2\rho} (\theta(r)^2 - \theta_0^2) + \frac{1}{2E\rho} (\sigma_r(r)^2 + 2\sigma_t(r)^2) + \frac{\alpha}{\rho} \theta(r) (\sigma_r(r) + 2\sigma_t(r)), \quad (109)$$

from which we obtain the internal energy associated to the initial state

$$I_i = 4\pi\rho \int_{r_1}^{r_2} \varepsilon_i(r)r^2 dr. \tag{110}$$

Using relation (40) and taking Eq. (108) into account, we get the entropy per unit mass η_i as a function of r , corresponding to the initial state

$$\eta_i(r) = C_E \ln\left(\frac{\theta(r)}{\theta_0}\right) + \frac{3E\alpha^2}{\rho}(\theta(r) - \theta_0) + \frac{\alpha}{\rho}(\sigma_r(r) + 2\sigma_t(r)). \tag{111}$$

In view of the Signorini theorem (see Gurtin, 1981, p. 113) the integral of $\sigma_r + 2\sigma_t$ over the container depends solely on p_1, p_2, r_1 and r_2 . Since external loads do not change passing from the initial state to the final state, the contribution of $(\alpha/\rho)(\sigma_r(r) + 2\sigma_t(r))$ to the total entropy is the same for the two states; thus, it is sufficient to consider the part of entropy related to temperature which, in view of Eq. (111), is

$$H_i = 4\pi\rho C_E \int_{r_1}^{r_2} r^2 \ln\left(\frac{\theta(r)}{\theta_0}\right) dr + 12E\alpha^2\pi \int_{r_1}^{r_2} r^2(\theta(r) - \theta_0) dr. \tag{112}$$

Now, let us consider the final state \mathcal{F}^e of container \mathcal{S}^e , which adiabatically reaches the equilibrium temperature $\bar{\theta}_e$, while maintaining constant pressures p_1 and p_2 , for the moment unknown. The corresponding stress field has components

$$\sigma_r^e(r) = -\frac{r_1^3 r_2^3 (p_1 - p_2)}{r_2^3 - r_1^3} \frac{1}{r^3} + \frac{p_1 r_1^3 - p_2 r_2^3}{r_2^3 - r_1^3}, \tag{113}$$

$$\sigma_t^e(r) = \frac{r_1^3 r_2^3 (p_1 - p_2)}{2(r_2^3 - r_1^3)} \frac{1}{r^3} + \frac{p_1 r_1^3 - p_2 r_2^3}{r_2^3 - r_1^3}, \tag{114}$$

and is independent of $\bar{\theta}_e$. The radial displacement associated to Eqs. (113) and (114) is

$$u^e(r) = \frac{r_1^3 r_2^3 (p_1 - p_2)}{2E(r_2^3 - r_1^3)} \frac{1}{r^2} + \left[\frac{p_1 r_1^3 - p_2 r_2^3}{E(r_2^3 - r_1^3)} + \alpha(\bar{\theta}_e - \theta_0) \right] r; \tag{115}$$

in particular, the displacements of the inner and outer boundaries are

$$u_{i1}^e = u^e(r_1) = \frac{r_1 r_2^3 (p_1 - p_2)}{2E(r_2^3 - r_1^3)} + \frac{p_1 r_1^4 - p_2 r_2^3 r_1}{E(r_2^3 - r_1^3)} + \alpha r_1 (\bar{\theta}_e - \theta_0), \tag{116}$$

$$u_{i2}^e = u^e(r_2) = \frac{r_1^3 r_2 (p_1 - p_2)}{2E(r_2^3 - r_1^3)} + \frac{p_1 r_1^3 r_2 - p_2 r_2^4}{E(r_2^3 - r_1^3)} + \alpha r_2 (\bar{\theta}_e - \theta_0). \tag{117}$$

The internal energy per unit mass ε_i^e as a function of r , corresponding to the state \mathcal{F}^e is

$$\varepsilon_i^e(r) = C_E(\bar{\theta}_e - \theta_0) + \frac{3E\alpha^2}{2\rho}(\bar{\theta}_e^2 - \theta_0^2) + \frac{1}{2E\rho}(\sigma_r^e(r)^2 + 2\sigma_t^e(r)^2) + \frac{\alpha\bar{\theta}_e}{\rho}(\sigma_r^e(r) + 2\sigma_t^e(r)), \tag{118}$$

from which we calculate the internal energy

$$I_1^e = 4\pi\rho \int_{r_1}^{r_2} \varepsilon_1^e(r)r^2 dr = \frac{4}{3}\pi\rho C_E(r_2^3 - r_1^3)(\bar{\theta}_e - \theta_0) + 2\pi E\alpha^2(r_2^3 - r_1^3)(\bar{\theta}_e^2 - \theta_0^2) + 4\pi\alpha(p_1r_1^3 - p_2r_2^3)\bar{\theta}_e + \mathcal{E}^e, \quad (119)$$

with

$$\mathcal{E}^e = \frac{2\pi}{E} \int_{r_1}^{r_2} (\sigma_1^e(r)^2 + 2\sigma_1^e(r)^2)r^2 dr. \quad (120)$$

Finally, the work L^e done by p_1 and p_2 passing from the initial state to the final one is

$$L^e = 4\pi r_1^2(u_{11}^e - u_{i1})p_1 - 4\pi r_2^2(u_{12}^e - u_{i2})p_2 = 4\pi\alpha(p_1r_1^3 - p_2r_2^3)(\bar{\theta}_e - \theta_0) + \mathcal{L}^e, \quad (121)$$

with

$$\mathcal{L}^e = 4\pi r_1^2 \left[\frac{r_1r_2^3(p_1 - p_2)}{2E(r_2^3 - r_1^3)} + \frac{p_1r_1^4 - p_2r_2^3r_1}{E(r_2^3 - r_1^3)} - u_{i1} \right] p_1 - 4\pi r_2^2 \left[\frac{r_1^3r_2(p_1 - p_2)}{2E(r_2^3 - r_1^3)} + \frac{p_1r_1^3r_2 - p_2r_2^4}{E(r_2^3 - r_1^3)} - u_{i2} \right] p_2. \quad (122)$$

We are now in a position to apply the first principle of thermodynamics, setting the variation in internal energy equal to the work of loads p_1 and p_2 ,

$$I_1^e = I_i + L^e. \quad (123)$$

By accounting for Eqs. (110), (119) and (121), relation (123) can be written as the second degree equation

$$F_1\bar{\theta}_e^2 + F_2\bar{\theta}_e + F_3 = 0, \quad (124)$$

whose coefficients F_1 , F_2 and F_3 have the following expressions

$$F_1 = 2\pi E\alpha^2(r_2^3 - r_1^3),$$

$$F_2 = \frac{4}{3}\pi\rho C_E(r_2^3 - r_1^3),$$

$$F_3 = \mathcal{E}^e - E_i - \mathcal{L}^e + 4\pi\alpha(p_1r_1^3 - p_2r_2^3)\theta_0 - 2\pi E\alpha^2(r_2^3 - r_1^3)\theta_0^2 - \frac{4}{3}\pi\rho C_E(r_2^3 - r_1^3)\theta_0. \quad (125)$$

Thus, $\bar{\theta}_e$ is the positive root of Eq. (124). The final entropy per unit mass as a function of r is

$$\eta_1^e(r) = C_E \ln\left(\frac{\bar{\theta}_e}{\theta_0}\right) + \frac{3E\alpha^2}{\rho}(\bar{\theta}_e - \theta_0) + \frac{\alpha}{\rho}(\sigma_1^e(r) + 2\sigma_1^e(r)). \quad (126)$$

As in the initial state, it is sufficient to consider the part of entropy related to the temperature which, from Eq. (126), is

$$H_f^e = \frac{4}{3}\pi\rho C_E \ln\left(\frac{\bar{\theta}_e}{\theta_0}\right)(r_2^3 - r_1^3) + 4\pi E\alpha^2(\bar{\theta}_e - \theta_0)(r_2^3 - r_1^3). \tag{127}$$

Now, let us consider the final state \mathcal{F}^m of container \mathcal{S}^m that with the same pressures p_1 and p_2 , adiabatically reaches the equilibrium temperature $\bar{\theta}_m$, which must be determined.

The radial stress (113) is purely compressive. On the other hand, there exist values of p_1 and p_2 such that the tangential stress (114) is positive starting at $r = r_1$, vanishes at a point $r^* \in [r_1, r_2]$ and becomes negative up to $r = r_2$. Thus, for certain values of p_1 and p_2 , if the material does not withstand tension, the stress field (113) and (114) does not represent a solution to the equilibrium problem. With a procedure similar to that used in Ref. (Padovani, 1996) it is possible, by starting out with the elastic solution (113) and (114) corresponding to a linear material, to calculate a negative semi-definite stress field equilibrated with the loads, which is therefore the solution to the equilibrium problem of \mathcal{S}^m . Such a stress field has components

$$\sigma_r^m(r) = \begin{cases} -p_1 \frac{r_0}{r^2} & r \in [r_1, r_0], \\ -\frac{r_0^3 r_2^3 (p_0 - p_2)}{r_2^3 - r_0^3} \frac{1}{r^3} + \frac{p_0 r_0^3 - p_2 r_2^3}{r_2^3 - r_0^3}, & r \in [r_0, r_2], \end{cases} \tag{128}$$

$$\sigma_t^m(r) = \begin{cases} 0, & r \in [r_1, r_0], \\ \frac{r_0^3 r_2^3 (p_0 - p_2)}{2(r_2^3 - r_0^3)} \frac{1}{r^3} + \frac{p_0 r_0^3 - p_2 r_2^3}{r_2^3 - r_0^3}, & r \in [r_0, r_2], \end{cases} \tag{129}$$

with $p_0 = p_1(r_1^2/r_0^2)$ and r_0 being the unique root belonging to the interval $[r_1, r_2]$ of the third degree equation

$$2p_1 r_1^2 r^3 - 3p_2 r_2^3 r^2 + p_1 r_1^2 r_2^3 = 0. \tag{130}$$

Thus, the spherical region \mathcal{P}_f , with inner radius r_1 and outer radius r_0 , exhibits radial fractures; while the remaining region \mathcal{P}_c , with inner radius r_0 and outer radius r_2 , is entirely compressed.

The radial displacement associated to Eqs. (128) and (129) is

$$u^m(r) = \begin{cases} \frac{r_1^2 p_1}{E} \frac{1}{r} + \frac{p_0 r_0^4 - p_2 r_0 r_2^3}{E(r_2^3 - r_0^3)} + \frac{r_0 r_2^3 (p_0 - p_2)}{2E(r_2^3 - r_0^3)} - \frac{r_1^2 p_1}{E r_0} + \alpha(\bar{\theta}_m - \theta_0)r, & r \in [r_1, r_0], \\ \frac{r_0^3 r_2^3 (p_0 - p_2)}{2E(r_2^3 - r_0^3)} \frac{1}{r^2} + \left\{ \frac{p_0 r_0^3 - p_2 r_2^3}{E(r_2^3 - r_0^3)} + \alpha(\bar{\theta}_m - \theta_0) \right\} r, & r \in [r_0, r_2]; \end{cases} \tag{131}$$

in particular, in place of Eqs. (116) and (117), we have

$$u_{f1}^m = u^m(r_1) = \frac{r_1 p_1}{E} + \frac{p_0 r_0^4 - p_2 r_0 r_2^3}{E(r_2^3 - r_0^3)} + \frac{r_0 r_2^3 (p_0 - p_2)}{2E(r_2^3 - r_0^3)} - \frac{r_1^2 p_1}{E r_0} + \alpha(\bar{\theta}_m - \theta_0)r_1, \tag{132}$$

$$u_{f2}^m = u^m(r_2) = \frac{r_0^3 r_2 (p_0 - p_2)}{2E(r_2^3 - r_0^3)} + \frac{p_0 r_0^3 r_2 - p_2 r_2^4}{E(r_2^3 - r_0^3)} + \alpha r_2 (\bar{\theta}_m - \theta_0). \tag{133}$$

The internal energy per unit mass ε_f^m , as a function of r corresponding to the final state \mathcal{F}^m is

$$\varepsilon_f^m(r) = C_E(\bar{\theta}_m - \theta_0) + \frac{3E\alpha^2}{2\rho}(\bar{\theta}_m^2 - \theta_0^2) + \frac{1}{2E\rho}(\sigma_r^m(r)^2 + 2\sigma_t^m(r)^2) + \frac{\alpha}{\rho}\bar{\theta}_m(\sigma_r^m(r) + 2\sigma_t^m(r)), \quad (134)$$

from which we obtain the internal energy

$$I_f^m = 4\pi\rho \int_{r_1}^{r_2} \varepsilon_f^m(r)r^2 dr = \frac{4}{3}\pi\rho C_E(r_2^3 - r_1^3)(\bar{\theta}_m - \theta_0) + 2\pi E\alpha^2(r_2^3 - r_1^3)(\bar{\theta}_m^2 - \theta_0^2) + 4\pi\alpha(p_1r_1^3 - p_2r_2^3)\bar{\theta}_m + \mathcal{E}^m, \quad (135)$$

where

$$\mathcal{E}^m = \frac{2\pi}{E} \int_{r_1}^{r_2} (\sigma_r^m(r)^2 + 2\sigma_t^m(r)^2)r^2 dr. \quad (136)$$

The work L^m done by the loads is

$$L^m = 4\pi r_1^2(u_{f1}^m - u_{i1})p_1 - 4\pi r_2^2(u_{f2}^m - u_{i2})p_2 = 4\pi\alpha(p_1r_1^3 - p_2r_2^3)(\bar{\theta}_m - \theta_0) + \mathcal{L}^m, \quad (137)$$

with

$$\mathcal{L}^m = 4\pi r_1^2 \left(\frac{r_0r_2^3(p_0 - p_2)}{2E(r_2^3 - r_0^3)} + \frac{p_0r_0^4 - p_2r_2^3r_0}{E(r_2^3 - r_0^3)} + \frac{r_1p_1}{E} - \frac{r_1^2p_1}{Er_0} - u_{i1} \right) p_1 - 4\pi r_2^2 \left(\frac{r_0^3r_2(p_0 - p_2)}{2E(r_2^3 - r_0^3)} + \frac{p_0r_0^3r_2 - p_2r_2^4}{E(r_2^3 - r_0^3)} - u_{i2} \right) p_2. \quad (138)$$

In view of the first principle of thermodynamics, by accounting for Eqs. (110), (135) and (137), $\bar{\theta}_m$ is a solution to the algebraic equation

$$F_1\bar{\theta}_m^2 + F_2\bar{\theta}_m + D_3 = 0, \quad (139)$$

where coefficients F_1 and F_2 are given in Eq. (125) and

$$D_3 = \mathcal{E}^m - E_i - \mathcal{L}^m + 4\pi\alpha(p_1r_1^3 - p_2r_2^3)\theta_0 - 2\pi E\alpha^2(r_2^3 - r_1^3)\theta_0^2 - \frac{4}{3}\pi\rho C_E(r_2^3 - r_1^3)\theta_0. \quad (140)$$

The final entropy per unit mass is

$$\eta_f^m(r) = C_E \ln\left(\frac{\bar{\theta}_m}{\theta_0}\right) + \frac{3E\alpha^2}{\rho}(\bar{\theta}_m - \theta_0) + \frac{\alpha}{\rho}(\sigma_r^m(r) + 2\sigma_t^m(r)), \quad (141)$$

and the final entropy related to temperature is

$$H_f^m = \frac{4}{3}\pi\rho C_E \ln\left(\frac{\bar{\theta}_m}{\theta_0}\right)(r_2^3 - r_1^3) + 4\pi E\alpha^2(\bar{\theta}_m - \theta_0)(r_2^3 - r_1^3). \quad (142)$$

By means of relations (72) and (73), we obtain the expression for the specific heat C_{E3} in the cracked

region \mathcal{P}_f , which is

$$C_{E3}(\theta) = C_E + \frac{2E\alpha^2}{\rho}\theta. \tag{143}$$

We observe that, since we have assumed the specific heat of the material subjected to purely compressive stress field to be constant, from Eq. (143) we deduce that when the material is cracked, its specific heat cannot be constant, but depends on temperature.

With the aim of comparing the final temperatures $\bar{\theta}_e$ and $\bar{\theta}_m$, we point out that

$$F_3 = D_3 + \Delta, \tag{144}$$

with

$$\Delta = \frac{\pi}{E} \left\{ -\frac{1}{r_2^3 - r_1^3} \left[2(r_1^3 p_1 - r_2^3 p_2)^2 + r_1^3 r_2^3 (p_1 - p_2)^2 \right] + \frac{2}{r_0} (r_0 - r_1) r_1^3 p_1^2 + \frac{1}{r_2^3 - r_0^3} \left(2p_1^2 r_1^4 r_0^2 - 6p_1 p_2 r_1^2 r_2^3 r_0 + \frac{r_1^4 r_2^3}{r_0} p_1^2 + 2p_2^2 r_2^6 + p_2^2 r_2^3 r_0^3 \right) \right\}. \tag{145}$$

For r_1, r_2 and p_1 fixed, Δ is a function of p_2/p_1 with domain $(r_1^2/r_2^2, (2r_1^3 + r_2^3)/3r_2^3)$. In particular, for $p_2/p_1 = (2r_1^3 + r_2^3)/3r_2^3$ we have $r_0 = r_1$ and $\Delta = 0$ and, consequently, $\bar{\theta}_e = \bar{\theta}_m$. Moreover, it is a simple matter to show that the limit of $\Delta(p_2/p_1)$ for p_2/p_1 going to r_1^2/r_2^2 is positive, and that Δ is a decreasing function of p_2/p_1 ; Δ is therefore positive. From Eqs. (124) and (139), by accounting for Eq. (144), we get the difference ω between $\bar{\theta}_e$ and $\bar{\theta}_m$

$$\omega = \bar{\theta}_m - \bar{\theta}_e = \frac{2\Delta}{(F_2^2 - 4F_1 D_3)^{1/2} + (F_2^2 - 4F_1 (D_3 + \Delta))^{1/2}}, \tag{146}$$

which is positive in view of the positiveness of Δ . From Eqs. (127) and (142) we obtain

$$H_f^m = H_f^e + \frac{4}{3}\pi\rho C_E (r_2^3 - r_1^3) \ln\left(\frac{\bar{\theta}_e + \omega}{\bar{\theta}_e}\right) + 4\pi E\alpha^2 (r_2^3 - r_1^3)\omega \tag{147}$$

and, by virtue of the positiveness of ω , we can conclude that

$$H_f^m \geq H_f^e. \tag{148}$$

Denoting by

$$\mathcal{D}^e = (H_f^e - H_i)\theta_0 \quad \text{and} \quad \mathcal{D}^m = (H_f^m - H_i)\theta_0, \tag{149}$$

the work lost in the process (Bailyn, 1994) by containers \mathcal{S}^e and \mathcal{S}^m , respectively, from Eq. (148), we get

$$\mathcal{D}^e \leq \mathcal{D}^m. \tag{150}$$

Fig. 3 illustrates the behaviour of ω as a function of the ratio p_2/p_1 varying in the interval $[1/\tau^2, (2 + \tau^3)/3\tau^3]$ in correspondence of $\tau = 2$ (continuous line), 3 (dotted line) and 4 (dashed line) with $\tau = r_2/r_1$. The graphs have been obtained using the following parameter values

$$E = 5 \times 10^9 \text{ Pa},$$

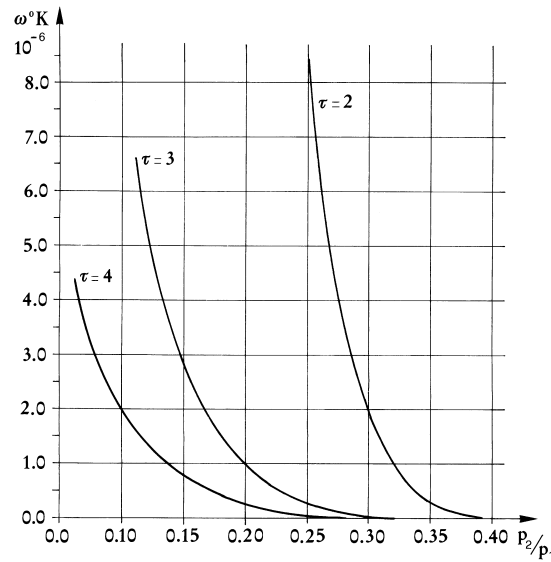


Fig. 3. ω vs. p_2/p_1 for different values of τ .

$$\begin{aligned}
 \nu &= 0, \\
 \theta_0 &= 303.15 \text{ K}, \\
 \theta_1 &= \vartheta_1 + \theta_0 = 325.15 \text{ K}, \\
 \theta_2 &= \vartheta_2 + \theta_0 = 293.15 \text{ K}, \\
 p_1 &= 3.4 \times 10^6 \text{ Pa}, \\
 \alpha &= 1 \times 10^{-5} \text{ K}^{-1}, \\
 C_E &= 1046 \text{ J/Kg K}, \\
 \rho &= 2000 \text{ Kg/m}^3.
 \end{aligned}$$

4. Conclusions

This paper has set forth a constitutive equation for no-tension isotropic materials in the presence of thermal expansion, as well as the basic coupled equations for thermoelastic equilibrium while accounting for the temperature dependence of material constants. In the absence of thermal variations, this constitutive model coincides to the model for no-tension materials.

Since this kind of constitutive equation allows one to explicitly calculate, not only the stress as a function of temperature and strain, but also the derivative of the stress with respect to the strain, it is particularly suited for use in a finite element code for solving equilibrium problems of no-tension solids subjected to thermal loads via the Newton–Raphson method (Padovani et al., 1999), in a manner analogous to that performed in Refs. (Lucchesi et al., 1994, 1995, 1996).

The simple example also presented demonstrates that the work lost in an adiabatic process is greater in a solid made of a no-tension material than in a solid made of a linear elastic material.

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Appendix A

Let us consider a plane stress situation and suppose $t_3 = \mathbf{q}_3 \cdot \mathbf{T} \mathbf{q}_3 = 0$. Denoting by \mathbf{E} the restriction of the strain tensor to the two-dimensional space orthogonal to \mathbf{q}_3 and by $e_1 \leq e_2$, the eigenvalues of \mathbf{E} , we define the following subsets of $\text{Sym} \times [\theta_1, \theta_2]$:

$$\mathcal{Q}_1 = \{(\mathbf{E}, \theta) \mid \nu(\theta)(e_1 - \beta(\theta)) + e_2 - \beta(\theta) \leq 0\}, \tag{A1}$$

$$\mathcal{Q}_2 = \{(\mathbf{E}, \theta) \mid \nu(\theta)(e_1 - \beta(\theta)) + e_2 - \beta(\theta) > 0, e_1 - \beta(\theta) \leq 0\}, \tag{A2}$$

$$\mathcal{Q}_3 = \{(\mathbf{E}, \theta) \mid e_1 - \beta(\theta) > 0\}. \tag{A3}$$

The expressions for the stress in the three regions are

if $(\mathbf{E}, \theta) \in \mathcal{Q}_1$, then

$$\mathbf{T}(\mathbf{E}, \theta) = \frac{E(\theta)}{1 + \nu(\theta)} \left\{ \mathbf{E} - \beta(\theta)\mathbf{I} + \frac{\nu(\theta)}{1 - \nu(\theta)} \text{tr}(\mathbf{E} - \beta(\theta)\mathbf{I})\mathbf{I} \right\}; \tag{A4}$$

if $(\mathbf{E}, \theta) \in \mathcal{Q}_2$, then

$$\mathbf{T}(\mathbf{E}, \theta) = E(\theta)(e_1 - \beta(\theta))\mathbf{q}_1 \otimes \mathbf{q}_1; \tag{A5}$$

if $(\mathbf{E}, \theta) \in \mathcal{Q}_3$, then

$$\mathbf{T}(\mathbf{E}, \theta) = \mathbf{0}. \tag{A6}$$

And finally, the expressions for free energy and entropy are

$$\psi(\mathbf{E}, \theta) = \xi(\theta) + \frac{E(\theta)}{2(1 - \nu(\theta)^2)\rho} \{e_1^2 + e_2^2 + 2\nu(\theta)e_1e_2 + 2(1 + \nu(\theta))\beta(\theta)(\beta(\theta) - e_1 - e_2)\}, \tag{A7}$$

for $(\mathbf{E}, \theta) \in \mathcal{Q}_1$,

$$\psi(\mathbf{E}, \theta) = \xi(\theta) + \frac{E(\theta)}{2\rho} (e_1 - \beta(\theta))^2, \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{Q}_2, \tag{A8}$$

$$\psi(\mathbf{E}, \theta) = \xi(\theta), \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{Q}_3; \tag{A9}$$

$$\begin{aligned} \eta(\mathbf{E}, \theta) = & -\xi'(\theta) - \frac{E'(\theta)(1 - \nu^2(\theta)) + 2\nu(\theta)\nu'(\theta)E(\theta)}{2\rho(1 - \nu^2(\theta))^2} \{e_1^2 + e_2^2 + 2\nu(\theta)e_1e_2 + 2(1 + \nu(\theta))\beta(\theta) \\ & \times (\beta(\theta) - e_1 - e_2)\} - \frac{E(\theta)}{2(1 - \nu(\theta)^2)\rho} \{2\nu'(\theta)e_1e_2 + 2\nu'(\theta)\beta(\theta)(\beta(\theta) - e_1 - e_2) + 2(1 \\ & + \nu(\theta))\beta'(\theta)(\beta(\theta) - e_1 - e_2) + 2(1 + \nu(\theta))\beta(\theta)\beta'(\theta)\}, \end{aligned} \quad (\text{A10})$$

for $(\mathbf{E}, \theta) \in \mathcal{Q}_1$,

$$\eta(\mathbf{E}, \theta) = -\xi'(\theta) - \frac{E'(\theta)}{2\rho}(e_1 - \beta(\theta))^2 + \frac{E(\theta)}{\rho}\beta'(\theta)(e_1 - \beta(\theta)), \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{Q}_2, \quad (\text{A11})$$

$$\eta(\mathbf{E}, \theta) = -\xi'(\theta), \quad \text{for } (\mathbf{E}, \theta) \in \mathcal{Q}_3. \quad (\text{A12})$$

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